Application of Fixed Point Theory and Dynamical System Theory to Collatz Conjecture

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Abstract

Let $\mathcal{N}$ and $\mathcal{R}$ be the set of all real numbers and the set of all positive integers, respectively. Then, the mapping $C(\cdot)$ on $\mathcal{N}$ with values in $\mathcal{N}$ is called Collatz mapping, if this mapping is defined as the following:

$$C(n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even,} \\ 3n+1, & \text{if } n \text{ is odd,} \end{cases}$$

where n means any positive integer, and, for any nonnegative integer k, k-time nested superposition of $C(\cdot)$, which is denoted by $C^k(\cdot)$, is defined as the following:

$$C^k(\cdots) = \begin{cases} I(\cdot), & \text{if } k \text{ is equal to } 0, \\ C(C^{k-1}(\cdot)), & \text{if } k \text{ is positive,} \end{cases}$$

where $I(\cdot)$ means the identity mapping.

In 1937, Collatz proposed the problem asking if, for any positive integer n, there exists a positive integer k_n satisfying $C^{(kn)}(n) = 1$. Nowadays, this conjecture is called Collatz conjecture, and there exist several mathematical approaches to this problem, which are based on various kinds of research areas such as number theory, probability theory and computation theory.

As is often the case with famous conjectures, this conjecture has also produced various conjectures which have been closely related to the original one. For example, we refer to the problem asking if there exists a positive integer n satisfying $\sup_{k\geq 0} C^k(n) = 1$, and moreover, the problem asking if there exists such an initial value m that $C^k(m) \neq 1$ holds for any $k \in \mathcal{N}$.

By the way, mathematical researches on the mappings which resemble the original Collatz mapping have been also promoted. For example, it is known that the freak

Collatz mapping $D(\cdot)$ on $\mathcal{N}$ with values in $\mathcal{N}$, which is defined as the following:

$$D(n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even,} \\ 3n - 1, & \text{if } n \text{ is odd,} \end{cases}$$

has at lease two periodic orbits which are different from each other. More concretely speaking, one periodic orbit $D(\cdot)$ runs through $\{1, 2\}$ and another one runs through $\{7, 20, 10, 5, 14\}$.

In this talk, we apply both the fixed point theory and the dynamical system theory to these conjectures which remain to be solved. Finally, all the presenters would like to express our hearty thanks to Professor Wataru Takahashi who has proposed the fixed-point-theoretic approach to this problem.